THE RICE INSTITUTE PAMPHLET

Vol. VIII January, 1921 No. 1

Plymouth Tercentenary Lecture

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THE RICE INSTITUTE

A university of liberal and technical learning founded by William Marsh Rice in the City of Houston, Texas, and dedicated by him to the advancement of Letters, Science, and Art

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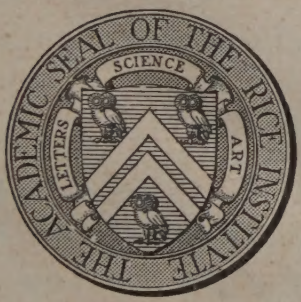

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I

EDUCATIONAL ENTERPRISES OF THE PILGRIM FATHERS[1]

THE mother and model of modern universities—the University of Paris—originated in a group of cathedral schools. These cathedral schools were founded by missions of a religion that made of hope a virtue. And similar missionary enterprises issuing from Rome established and endowed the early schools of England and Scotland, whose influence later extended in turn to the new continent across the seas: from Cambridge through Emmanuel and Magdalene to Harvard; from Edinburgh through the Scottish Presbyterians to Princeton. The same translated ancient Anglo-Saxon influences appear in the eighteenth-century beginnings of King's College, the forerunner of Columbia University in the North, and in the early nineteenth-century history of the University of Virginia, a history that has been subsequently significant to all succeeding institutions in the South; but the seventeenth century, that witnessed the founding of colleges in Williamsburg, Virginia, and Cambridge, Massachusetts, was preëminently the Anglo-Saxon century of America.

PILGRIM TERCENTENARY

THE backgrounds I have just sketched are brought into the foreground of our thinking this autumn by the approaching tercentenary of the Pilgrim Fathers, for it was on December 21, 1620, that John Robinson's flock, after sixty-six

[1] Several paragraphs from the Matriculation Address of the ninth academic year of the Rice Institute, delivered Wednesday morning, September 22, 1920.

days of storm-bound passage across the Atlantic and a month in the vicinity of Provincetown Harbor, though originally destined for a more congenial southern shore, landed its first scouting-group on Plymouth Rock, which, despite its modest dimensions, must have loomed up literally as "the shadow of a great rock in a weary land." Nor can I ever think of that rock without recalling a revered University of Virginia preceptor of mine, who, contrasting the Cavalier colonists of Virginia with the Puritan pioneers of New England, said, and often said, what a pity it was that when the Pilgrim Fathers landed on Plymouth Rock it had not been a case of Plymouth Rock landing on the Pilgrim Fathers. In anticipation of the formal anniversary celebrations that will be held throughout the country in December of this year, there is coming to America this autumn a mission of Pilgrims, among them Sir Arthur Everett Shipley, Master of Christ's College, Cambridge University. Sir Arthur will, upon our invitation, be delivering at the Rice Institute next month not only a Pilgrim Tercentenary lecture, but also a course of public lectures on several subjects in the history of science. He is pleasantly remembered among us as the Head of the British Educational Mission to the Universities of the United States, which paid the Rice Institute a memorable visit during Thanksgiving week of 1918. That visit has also been happily recalled to us lately by an article in the *New York Evening Post* of July 31st last, in which a member of the party is quoted at length from the *Landmark,* a journal of the English-speaking Union, where, among other things in advocacy of the interchange of students between British and American universities, the writer says: "Americans must come to other British universities (not alone to Oxford and Cambridge)—to Manchester, Dublin, Glasgow, Sheffield, Leeds. And Englishmen must be told more and

more, he continues, "that it is to their own advantage, and to the advantage of the world, that they should go to America—not only to Harvard and Yale and Princeton, but to Minnesota, Wisconsin, Cornell, California, and the Rice Institute."

PROVISION FOR HIGHER EDUCATION

For this institution, founded by a native son of Massachusetts, perhaps the most striking thing to be recalled by the three hundredth anniversary of the founding of Plymouth Colony, is that the immediate concern of the colonists, after making primitive provision for their physical necessities, was the founding of a college. "After God had carried us safe to New England," says a contemporary document, "and we had builded our houses, provided necessaries for our livelihood, reared convenient places for God's worship, and settled the civil government, one of the next things we longed for and looked after was to advance learning and perpetuate it to posterity, dreading to leave an illiterate ministry to the churches, when our present ministers shall lie in the dust. And as we were thinking and consulting how to effect this great work, it pleased God to stir up the heart of one Mr. Harvard (a godly gentleman and a lover of learning, there living amongst us) to give one half of his estate (it being in all about £1700) towards the erecting of a college, and all his library. After him another gave £300, others after him cast in more, and the public hand of the state added the rest." The part contributed by the public hand of the state, to which reference is here made, was authorized by the general court of the colony, which in 1636 passed a resolution agreeing "to give £400 towards a school or college, whereof £200 shall be paid next year, and £200 when the work is finished, and the next court to appoint where and what build-

ing." The passing of this resolution in 1636 Harvard commemorates as its act of foundation, and such was the beginning of the oldest college on the American continent.

PRINCIPLES UNDERLYING PUBLIC EDUCATION

MOREOVER, by these same Pilgrim pioneers, in the Massachusetts Acts of 1642 and 1647, compulsory education as a righteous demand on the state and on the part of the state was affirmed. In substance they affirmed that "the universal education of youth is essential to the well-being of the state; that the obligation to furnish this education rests primarily upon the parent; that the state has the right to enforce this obligation; that the state may fix a standard which shall determine the kind of education and the minimum amount; that public money raised by general tax may be used to provide such education as the state requires, and the tax may be general though the school attendance is not; that education higher than the rudiments may be supplied by the state, and opportunity must be provided at the public expense for youths who wish to be fitted for college." On these half dozen underlying principles rests the foundation of public school education in the United States as developed from state to state, but it is only within the last fifty years that legislation has been provided in England, first in 1870 and finally in 1918, effective in 1920, for a national system of public education based on such principles as were operative in an American British colony nearly three hundred years ago.

SPIRIT OF PILGRIM MOVEMENT

THIS is neither the time nor the place to commemorate adequately or at length the Puritan movement. It was literally a crusade for righteousness and a Pilgrim's Progress to a promised land. In austere devotion to the duty of life

it missed a great deal of the joy of life, and to us the joy of life is no less a duty than the duty of life. It had great faith—faith in heaven, faith in man, faith in work. It had great hope, "the hope that hopes till it creates the thing it contemplates." It had great zeal, sometimes frenzied to the fanaticism of untempered zeal. It had great loyalty, loyalty to law and loyalty to liberty. And to the Puritan liberty meant religious liberty, and law the word of God. To him the great purpose of life was the glory of God and the service of man: to these ends he sought to Christianize and educate the Indian rather than crush and exterminate him. For him philanthropy was synonymous with the salvation of souls, and he began with a passion for saving his own soul. With the Plainsman of a later day, the resolute loyalties of his patriotism were reserved for the freedom of self-government, even in the wilderness, if of his own choosing, and human progress for him meant literally human pilgrimage from this world to a better world to come, eternal in the heavens. These were the enthusiasms of his spirit, these the larger freedoms of his mind. And he entered on them with undoubting assurance of providential guidance.

II

Plymouth Tercentenary Lecture

WHAT THE PILGRIM FATHERS LEFT BEHIND THEM[1]

ON this side of the Atlantic we are pretty well informed as to what the Pilgrim Fathers found on their arrival. It may be, however, worth while to consider for a time what they left behind them.

Much they abandoned certainly would not have been appreciated or even known to them. They were almost wholly plain country folk, used to husbandry and life on the land. They sought freedom in Holland, and found it, but not happiness. Still were "Abana and Pharpar, rivers of Damascus, better than all the waters of Israel." They agreed among themselves "how grievous it was to live from under the protection of the State of England," and "how like we were to lose our language, and our name, of English." And, since the England of James I would have none of them, save at a price they would not pay, they set out in quest of "some corner of a foreign field" which they might make "for ever England." One of the reasons why they left Holland after a ten years' sojourn was that in order to earn a living they were compelled to practise the arts of the factory. Only by intense mechanical and indoor labor and by the use of an intensive child labor were they able to get a bare living in Holland. Further, they wished to remain

[1] A lecture in commemoration of the tercentenary anniversary of the founding of Plymouth Colony, delivered at the Rice Institute October 14, 1920. Parts of this address appeared subsequently in *The Outlook* (New York), and certain other parts in the *Saturday Magazine* of the *New York Evening Post.*

English folk and to be under the English king, though they were as anxious as anybody to put a quite considerable distance between themselves and His Majesty.

Much that they left behind them was bad: perpetual strife between the countries of Europe; strife in the church and between the churches. Superstition was as rampant then as it is to-day; but then it was accompanied by far more cruelty. Under the Tudors the laws against witches were milder than in other countries; but under James I these laws were repealed, and he himself took an active part in the cruel and senseless persecution. "From Witches, Warlocks, and Wurricoes, an' evil Spirits an' a' Things that gang bump i' the Nicht . . . Guid Lord deliver us," was a frequently repeated prayer from the Scottish litany. On an average nearly a thousand men and women were annually done to death for alleged witchcraft in the first half of the seventeenth century.

The first fifth of this century, 1600–1620, in Europe was a period of intense interest in every form of human activity. At its beginning, Queen Elizabeth—like all the Tudor monarchs most highly educated—was on the throne of England. She was succeeded in 1603 by James VI of Scotland and first of England—the most learned fool in Christendom. Robert Cecil was still chief adviser to the English crown. Henry IV, the first of the Bourbons, was on the throne of France, and before our period was complete, Richelieu was taking command of the policies of his comparatively insignificant son, Louis XIII. The martial Julius II, the della Rovere Pope, who began to build St. Peter's, sat in the Apostle's Chair in Rome. He was succeeded by Leo XI, the last of the series of Medici Popes.

As usual, half the countries of Europe were more or less at war. England was fighting Ireland, who had the sup-

port of Spain, which had at Lisbon (1599) equipped a second Armada, turned to naught at its first sailing by a storm. The Poles were fighting Russia and had taken Moscow. About the middle of our period the first of the Romanoffs, Michael, son of Philaret, patriarch of Moscow, became czar and founded the unhappy and tragic line of Romanoffs. Spain was fighting for the Netherlands. The Thirty Years' War, which for three decades devastated Central Europe, was commenced in Bohemia by the act of the Bohemian nobles, led by Thurn. They had revolted and hurled the two regents from a window of the palace at Prague in 1618. In the following year Maurice the stadholder, who was warring with the Holy Roman Empire (which, as somebody—probably Lord Bryce—has pointed out, was neither holy, nor Roman, nor an empire), was executing Barneveldt and imprisoning Grotius in the castle of Louvestein. The Hungarians were fighting and annexing first Transylvania and then Moldavia; Savoy was trying to annex Geneva, and so it all went on. Some of these incessant wars were due to the ambitions of the several rulers for more territory, some were due to religious antinomies, many owed their being to a combination of the two. One doubts if our Pilgrim Fathers, first at Amsterdam, then at Leyden, knew much about it all, or if they knew, whether they cared. They had other things to think about.

It was a period of great commercial activity. In 1600, owing to the increase in the price of pepper by the Dutch, whose fault has ever been in "giving too little and asking too much," an association of London merchants, with 125 shareholders and a capital of £70,000, was formed for trading with the East Indies. The numerous Dutch companies trading in the East amalgamated two years later into the Dutch East Indian Company, and, after ejecting

the Portuguese from the Moluccas, they monopolized the spice trade. Two years later, Henry IV sent De Monts to colonize Acadia, and Annapolis, then called Port Royal, was founded. Champlain was exploring the western coast of North America. The following year the Barbadoes, "the first British Colony," was taken by the British, but not "settled" until 1624, and in 1607 John Smith was starting a settlement at Jamestown, in the south of Virginia. Many years later Captain John Smith, the hero of the Pocahontas story, offered his services to the Pilgrim Fathers, but they were declined, and it is with a certain complacency that the inveterate soldier of fortune tells us how "their humorous ignorances caused them for more than a year to endure a wonderful deal of misery with an infinite patience." In 1608 Champlain founded Quebec and began his protracted struggle with the Iroquois, and next year Paraguay was handed over by Spain to the Jesuits, who established there a theocracy based on communism. About the same time the Bermudas were annexed by the Virginia Company, and a colony was first planted there in 1612. Champlain was in 1615 exploring Lake Huron, whilst the year before the United New Netherland Company, recently established in Holland, received territories at the mouth of the Hudson.

The intense desire to find a shorter way to China than that around the Cape was the cause of much exploration during our period. Henry Hudson made no less than four attempts to get round the north of America between the years 1607 and 1611, on the third of which he made his way one hundred and fifty miles up the river which bears his name. At the same time Champlain was coming down from the northern lakes—in fact, these two explorers approached within twenty leagues of each other. In 1610 he penetrated into Hudson's Bay, at once his monument and his grave.

Some of these activities on the eastern shores of America may have been known to the leaders of the Pilgrim Fathers, but one doubts if they knew or would have cared to know about the great changes which were taking place in the eastern world. The Manchu Tartars were invading China and in the year of the Pilgrims' voyage proclaimed their independence of that country. The Sikhs were fighting a holy war against the Mogul emperors. The Dutch, about the same time, were buying the Island of Goree and establishing Batavia in Java, and a British company was chartered to trade with West Africa and establish forts on the Gambia and the Gold Coast. In the East the Dutch had just obtained permission to trade with Japan, whilst the British at the same time were making a settlement at Surat, near Bombay, under the auspices of Sir Thomas Roe, Ambassador to Jehangir, the son of the great Mogul emperor Akbar.

In 1616 the cultivation of tobacco was introduced into Virginia, probably much against James I's wishes, and three years later the first colonial parliament, that of South Virginia, met at Jamestown. This was the first constitutional, free-elected parliament in America, and but for the fact that a similar but naturally much smaller institution had been established in the Bermudas a few months before, it would be reckoned second in point of time to the House of Commons in England. In the same year negro slaves were brought to this "Plantation."

William Baffin, a year or two later, explored the great inlet, afterward called by his name. Later he went east and is said to have been mate in a ship voyaging to Surat and Mocha. It is believed he was killed while helping the Persians to expel the Portuguese from Ormuz.

A distinguished Spanish explorer, Torres, in 1606, was sailing between New Guinea and Australia through what we

now call the Torres Strait. In 1614 Pietro della Valle was starting on his journey through Syria, Persia, and India.

The world was shrinking.

The Pilgrim Fathers left behind them great heritages of art, literature, and science. They were nearly all of them young men, few of them on the "shady side" of thirty-two, and the twenty years at the beginning of the seventeenth century were the formative years of their lives. Still one doubts whether they had any real appreciation of even the Dutch and Flemish art, which they can hardly have escaped seeing. Guido Reni, who died the same year as Galileo (1642), was painting in Rome, and Rubens had returned from his seven years' study in Italy and had settled at Antwerp to become court painter to the Archduke Albert. He completed his "Descent from the Cross" in 1612. Murillo and Rembrandt were born in our period, and Van Dyck, Goyen, and Frans Hals were young men. Inigo Jones was designing his magnificent palace at Whitehall after the manner of Palladio; only the banqueting hall was carried out. The most beautiful part of the Schloss at Heidelberg, the Friedrichsbau, was completed in 1607; Velasquez was reaching the crowning point of his career and was shortly to be asked to settle in Madrid and to accept the appointment of court painter.

The first oratorio, composed by Cavaliere, was performed in the Oratory at Rome in 1600, and at the same time "Eurydice," the libretto by Rinuccini, the music by Peri, was performed at the marriage of Henry IV and Mary de Medici. From it came modern opera.

Ben Jonson was publishing his "Volpone" and beginning to compose plays and masques with music and scenery, which remained popular at the court until the Puritans cleared all this sort of thing out of the country.

Shakespeare's Sonnets appeared without his sanction in 1609. "Hamlet," published seven years previously, was the first of his greater plays, the remainder of which were written in the nine succeeding years.

The two supreme glories of the English tongue are "Mr William Shakespeare's Comedies, Histories and Tragedies" and the wonderful translation of the Bible ordered by King James I in the early part of our period. This was published in 1611 and has ever been known as the Authorized Version, which found its way to the hearts of English people as no other book has ever done, and we may be sure it found its way to the hearts of the Pilgrim Fathers.

Milton at the time of the Pilgrim Fathers had just been painted by Cornelius Janssen as "a boy of ten." He made a charming picture of a serious-looking but charming boy.

Stow had just completed his "Survey of London," Coke was issuing his "Law Reports," and Casaubon his "Commentaries." "Don Quixote," the chief masterpiece of Cervantes—one of the great writers of all time—appeared within our period, and Lope de Vega was then publishing his pastoral novels and his poems. Calderon was born with the century.

The same year that Cervantes published his "Don Quixote" saw the birth of Francis Bacon's "Advancement of Learning," and his still greater "Novum Organum" was printed in the very year of the sailing of the Pilgrim Fathers. Harvey, who was working when Bacon was writing, said of him, "He writes philosophy like a Lord Chancellor." This, perhaps, is true, but his writings show him a man weak and pitiful in some respects, yet with an abiding hope, a sustained object in life, one who sought through evil days and in adverse conditions "for the glory of God and the relief of man's estate."

Captain John Smith was almost beginning American literature by the publication of the "True Relation of Virginia"; Donne, the melancholy and certainly morbid dean of St. Paul's, was publishing his "Anatomy of the World" and his "Satires"; Robert Burton was publishing his "Anatomy of Melancholy," which in later editions he greatly enlarged.

It was also the time for the establishment and inception of many learned and scientific societies and academies. In 1603 Cesi established the Academia dei Lincei in Rome, and four years later the Lutherans deserted Marburg in Hesse and founded in the same state a rival university at Giessen. The Ambrosian Library at Milan was started in 1609 by Cardinal Federigo Borromeo. Francis de Sales, who in 1608 had published his "Introduction to a Devout Life," with Madame Chantal founded in 1610 the female Order of the Visitation, modeled on the Ursulines, which spread with great rapidity and met with the Pope's approval. The establishment of the Fruchtbringende Gesellschaft in Weimar on the lines of the Italian societies made literary circles in Germany popular. Madame de Rambouillet began to form a circle of litterateurs which dominated French taste for a generation. The Accademia della Crusca was issuing its Dictionary, and various experiments were being made with education.

All these activities, however, would have left the Pilgrim Fathers unmoved: they were beyond the sphere of their vision.

But great as were the first years of the seventeenth century in art and in literature, it was equally great in every branch of science. In 1613, the New River, still the source of much of London's water supply, was brought into the city by Sir Hugh Myddleton. That at any rate they would have appreciated and understood.

In 1600 Gilbert published his "De Magnete," the first considerable contribution to British science since those made by Roger Bacon nearly four centuries before. Kircher invented the magic lantern, and the great Danish astronomer, Tycho Brahe, died in the first year of our century. Kepler (1571–1630) was explaining the structure of the eye and how images of objects are formed on the retina. He was also throwing much light on rainbows, tides, and the motion of the planets. In 1608 Lippersheim, about whom little or nothing seems to be known, invented the telescope, and the next year either Galileo or Joannides, or both, invented the microscope. There are very few clear-cut inventions in the world,—most are the result of successful efforts of many inventors striving after a common end. The telescope was greatly improved by Jansen, a Dutch spectacle-maker, and by Galileo, who applied it to astronomy; for the first time the four satellites of Jupiter, the mountains in the moon, Saturn's rings, sun spots, and the various phases of Venus were seen by the human eye. Galileo determined the period of the revolution of the sun on its axis and confirmed his faith in the Copernican system, which was indeed adopted by the learned of the time, although the world at large accepted the Ptolemaic system—a system which, as a schoolmaster, Milton taught. Mark Pattison has pointed out that these two systems "confront each other" in Milton's poems in much the same relative position which they occupied in the mind of the public. "The ordinary, habitual mode of speaking of celestial phenomena is Ptolemaic; the conscious or doctrinal exposition of the same phenomena is Copernican." As is well known, Galileo got into trouble with the church, who, after all, did not treat him very harshly. The Pope had to take official cognizance of the heresy as Pope; but he seems to have been a kindly old

gentleman—at any rate, according to the story which relates that when Galileo told him the earth moved round the sun, he replied, "That is all very well, but what are you going to do about it?" However, this may be as apocryphal as the classical words put in the astronomer's mouth on the occasion of his recantation before the inquisitor in the church of Santa Maria sopra Minerva, *"E pur si muove."*

Napier of Merchiston had made known his discovery of logarithms and his friend Briggs published the first tables in 1617. A few years later the slide rule, which to-day plays a great part in physical and engineering science, was invented by Edmund Gunter. Algebra was being written in the notation we still employ, and decimals, with their "damned little dots," as Lord Randolph Churchill described them, were coming into use. At the end of our period Drebbel constructed a thermometer, employing spirits of wine in his bulb, and Bacon was suggesting that heat might be a form of motion. About the same time a certain Dutchman named Snell discovered the law of refraction of light and calculated its index for water and other substances.

On the biological side great discoveries were being made. Fabricius in 1603 discovered and described the valves in the veins, and nineteen years later Assellius was for the first time describing the lacteal system. But most important of all, before the end of our period, Harvey was lecturing in St. Bartholomew's Hospital on his discovery of the circulation of the blood. Harvey, "the little choleric man," as Aubrey calls him, was educated at Cambridge and at Padua, and was in his thirty-eighth year when, in his lectures on anatomy, he expounded his new doctrine of the circulation of the blood to the College of Physicians, although his *Exercitatio* on this subject did not appear till 1628. His notes for the lectures are now in the British Museum. He was

physician to Charles I, and it is on record how, during the battle of Edgehill, he looked after the young princes as he sat reading a book under a hedge a little removed from the fight.

In the chain of evidence of his convincing demonstration of the circulation of the blood, one link only, to be supplied by the invention of the compound microscope, was missing. This, the discovery of the capillaries, was due to Malpighi, who was among the earliest anatomists to apply the compound microscope to animal tissues. Still, as Dryden has it,

> The circling streams once thought but pools of blood—
> (Whether life's food or the body's food),
> From dark oblivion Harvey's name shall save.

Harvey was happy in two respects as regards his discovery. It was, in the main and especially in England, recognized as proven in his own lifetime, and, again, no one of credit claimed or asserted the claims of others to priority. In research, all enquirers stand on steps others have built up; but in this, the most important of single contributions to physiology, the credit is Harvey's and almost Harvey's alone. Cowley, a man of wide culture, wrote an "Ode upon Dr. Harvey," in which his achievement was contrasted with a failing common to scientific men of his own time, and, so far as we can see, of all time:

> *Harvey* sought for Truth in Truth's own Book
> The Creatures, which by God Himself was writ;
> And wisely thought 'twas fit,
> Not to read Comments only upon it,
> But on th' original it self to look.
> Methinks in Arts great Circle others stand
> Lock't up together, Hand in Hand,
> Every one leads as he is led,
> The same bare path they tread,

A Dance like Fairies a Fantastick round,
But neither change their motion, nor their ground:
Had *Harvey* to this Road confin'd his wit,
His noble Circle of the Blood, had been untroden yet.

As we have seen, the Pilgrim Fathers left all these things behind them: "the rumours and the marching and the strife"; unparalleled development in science, in literature, and in art. It is more than probable they knew not what they left. Much of the activity of the first twenty years of the seventeenth century was taken up with religious and civil controversies and contentions; to these we have not alluded, for the Pilgrim Fathers did not leave these behind—they took them with them.[1]

The Pilgrim Fathers crossed in the *Mayflower:*

I think some angel christened her,
Touched her black bows with dew and flame,
And watched her through the sunset bear
The light of England's loveliest name.

O little fragrant stars of snow
That bloom in England, laughing May,
The sea wind wafts your scent to-day
Across three thousand miles of spray.

From the "Mayflower," by Alfred Noyes.

We must never forget that the Pilgrim Fathers were young men—few of them were on the shady side of thirty-two—and all young men have something of the spirit of adventure in them. With the exception of that soldier of fortune Miles Standish, who was thirty-six, they were all born in the year (1588) of the Spanish Armada or thereabouts, and as children must have heard much about it.

[1] For many of the facts and dates in the above pages I am indebted to "Annals of Politics and Culture," by G. P. Gooch, M.A., Cambridge University Press, 1901.

Their boyhood was spent in one of the most stirring times in the history of England. Queen Elizabeth's reign was drawing to a close, but the adventurous spirit of the great Elizabethans still persisted.

They were not the only people who had left Europe in the hope of obtaining religious liberty in the New World.

Froude tells us that "a colony of Huguenot refugees had settled on the coast of Florida. The Spaniards heard of it, came from St. Domingo, burned the town, and hanged every woman and child, leaving an inscription explaining that the poor creatures had been killed not as Frenchmen, but as heretics. Dominique de Gourgues of Rochelle heard of this fine exploit of fanaticism, equipped a ship, and sailed across. He caught the Spanish garrison, which had been left in occupation, and swung them on the same trees—with a second scroll saying that they were dangling there not as Spaniards, but as murderers."

The great colony of Virginia, ever associated with Kingsley's great novel "Westward Ho!" was founded by folk who frankly acknowledged adventure and commercial aims as the chief objects in their enterprise, but the Pilgrim Fathers—and by Pilgrim Fathers I mean those that crossed in the *Mayflower,* and not their more intolerant successors, the Puritans—mingled a certain spirit of adventure and speculation in their historic voyage for religious liberty. But they meant to stay: they were no birds of passage. They brought their wives and children with them. They were real colonizers.

Certainly many of their supporters took a great interest in establishing English dominion and English trade in the temperate regions of North America. If the claims of Britain to the temperate regions of North America were to be made good, it was needful that some effective settlement

of English folk should be established there. Captain John Smith, the navigator, saw this clearly. He knew that Dutch and French traders were casting eager eyes on the fair territories which he fondly named "New England." But he found it hard to arouse interest at home. "Nothing," he says, "would be done for a Plantation till about some hundred of your Brownists of England, Amsterdam, and Leyden went to New Plymouth." There, in the first year, he goes on to say, they endured a "wonderful deal of misery with an infinite patience . . . but those in time doing well, divers others have, in small handfuls, undertaken to go there to be several lords and kings of themselves; but most vanished to nothing."

Plymouth Plantation, however, did not vanish. The Pilgrims succeeded in showing that hard work and intelligent management could wring a fair living from an inhospitable soil. At first they started on a wrong basis. Like the Soviet Government of Russia to-day, they believed in the community of goods; still, they were intelligent men, and it took them little more than a year and a half to learn "that the doctrine of 'no private property' was impracticable even in a community bound together by religious convictions." To this day "people of American stock believe heartily in private property, individual and family thrift, and the transmission of savings to descendants. The teachings of Lassalle and Marx therefore take no hold on Americans."

They certainly had something of the spirit of Sir Walter Raleigh in them. Raleigh, who broke his heart and lost his head over the matter, never failed to believe that a great British kingdom would arise on the western shores of the Atlantic. "I yet shall see her an English nation," wrote Raleigh as he sat in his cell in the Tower of London. Many of his adventures went awry. His colony on Roanoke

Island, North Carolina, collapsed, and it was Drake who brought its few survivors back to England. It is interesting to recall that Drake subsequently became member of Parliament for Plymouth, and without knowing much about the intervening members of that seaport borough, we may recall that one of the present members, the first woman to sit in Parliament, Lady Astor, is a native of the neighboring State of Virginia.

Throughout the recent celebrations in memory of the *Mayflower* stress has been laid, and rightly laid, on the religious motive of the Pilgrim Fathers' great adventure. Conceding this to be the primary stimulus, there yet remain others. At the beginning of the seventeenth century men were becoming emancipated; no longer did they find security in the written words of the ancients. This emancipation impelled men to seek new outlets, and it was aided and enforced by the commercial requirements of the time. Everywhere exploration was going on. The French, the Spaniards, the Italians, the Portuguese, and the English were seeking new worlds to conquer. The earth was growing smaller.

Too much stress must not be laid—as the clergy are apt to lay it—on the purely religious principles which impelled our Pilgrims to make their great break with their life in the Old World. Still, the fact that they came not entirely for gain played a large part in their success as colonizers. Not only had they the high spirit of enterprise, but they had the sober, constant quality of enduring industry. To be a successful colonist one must not only have the light-hearted adventurous spirit of Hawkins, Drake, and Raleigh, but a deep and abiding seriousness of purpose. This the Pilgrims had, and in spite of persecution and affliction their stern qualities enabled them to endure to the end, and if they were

intolerant, they were less so than their successors, the Puritans. They supplied that element in New England which is expressed by the Englishman who has described the New Englander as the "east wind made flesh."

The fact that their landing place was named for their port of sailing, a name first suggested by Charles I, then Prince of Wales, has given rise to the erroneous idea that many of the Pilgrim Fathers came from the west of England. This is not so. More than 75 per cent. came from the eastern side of England—32 from Norfolk, 17 from Kent, 17 from the City of London, 11 from Essex, and from the north of England came the balance. There were, indeed, no westerners among their band, and what perhaps is still more remarkable, there were no Jews, no Scotch, and no Irish. They were, in fact, of pure English extraction. Three of them were fellows of Cambridge colleges, but it does not appear that Oxford contributed in any way to their number.

In conclusion, we must pay a brief tribute to the Pilgrim Mothers. Their sufferings and their courage must at least have equaled that of their husbands. They had to endure the grumbling of the Fathers—think of the food!—and the wailing of the children, and, in their cramped quarters, had to humor and to comfort both.

Some of the discomforts and perils of the voyage are recorded by Governor Bradford:

"These troubles being blown over, and now all being compact together in one ship, they put to sea again with a prosperous wind, which continued divers days together, and was some encouragement to them. Yet, according to the usual manner, many were afflicted with seasickness. . . .

"After they had enjoyed fair winds and weather for a season, they were encountered many times with cross winds,

and met with many fierce storms, with which the ship was shrewdly shaken, and her upper works made very leaky. . . .

"In sundry of these storms the winds were so fierce and the seas so high as they could not bear a knot of sail, but were forced to hull [drift] about without canvas set for divers days together. . . ."

Still, we may well believe that each of them said in her heart what another Pilgrim lady said in another Pilgrim's Progress:

> Bless'd be the day that I began
> A Pilgrim for to be;
> And blessed also be that man
> That hereto movèd me.

A. E. Shipley.

III

GRAVITATION AND THE ETHER

EINSTEIN'S theory of gravitation may be compared with thermodynamics. Both start with a few fundamental principles or axioms based on experience and proceed by logical deductions without further appeal to facts. Neither attempts any explanation of the nature of the phenomena considered.

The kinetic theory of gases is an example of a different type of theory. A definite mechanism is assumed for a gas and the properties of the mechanism are worked out and compared with the known properties of gases.

If the axioms of the first type of theory are true, the theory itself must be true, unless the reasoning is at fault; both types of theory are to be tested by comparison of the results with observation.

The principles on which Einstein's theory depends are his principle of relativity and his principle of equivalence. According to the first, the laws of nature must be such that the differential equations which express them are independent in form of the coördinate system or frame of reference used, and, according to the second, a gravitational field cannot be distinguished from an apparent field due to an acceleration of the frame of reference. From these principles he shows how the form of the paths or orbits described by bodies moving in a gravitational field may be deduced.

The theory is mainly geometrical. It gives the geometrical form of the orbits. In the absence of a gravitational

field a body moves in a straight line, provided the frame of reference is suitably chosen, and in the field surrounding a heavy particle a small body describes an ellipse, as in Newton's theory, again provided the frame of reference is suitably chosen. In Newton's theory the ellipse remains stationary, while in Einstein's it slowly rotates in its own plane.

What causes the moving particle to move round an ellipse instead of along a straight line? According to Newton, there is an attraction between the particles, a force tending to pull them together. But this is merely another way of saying that they tend to move together; it is no explanation. If we regard Einstein's geometry of orbits as a geometry of space, we may say that the body describes an ellipse because the space and time around a heavy particle are not straight but curved. The path of the particle is the natural path in the curved space-time: the path of maximum interval length. This does not seem to be an explanation any more than Newton's force. Why should a particle describe a curve in a curved space, and, anyhow, what is the meaning of curved space?

The idea of space is obtained by experience; it cannot be explained. We express our experience of space by saying that bodies occupy parts of space and can move about in it; between bodies there is what is usually said to be empty space, or a vacuum. Such statements seem to be intelligible because of our experience of bodies and their motions; they would mean nothing to a being without experience similar to our own. To explain what is meant by curved space or non-Euclidean space it is customary to take the case of a surface which may be flat or curved. If we consider a sphere on which a particle is constrained to move, then we see that this particle will not move in straight lines, but along great circles of the sphere. In the same way, a par-

ticle in curved space of three or four dimensions, it is said, will move along a curve. The objection to this illustration is obvious. The sphere must be a material sphere, otherwise it could not keep the particle on its surface. There must be something curved to make a curved space. If, then, the space between bodies is empty, there is nothing in it to be curved, so it cannot be curved.

Suppose that a large number of cubical blocks were made at a certain place, all exactly equal when tested by placing any face of one against any face of the others. Then suppose these blocks were taken into the gravitational field near a heavy particle and were built up by placing them close together so as to fill up the space around the heavy particle. Suppose it was then found that they would not fit together exactly, so that the space could not be filled up completely with them. Ought we to conclude that the space was curved or that the shape of the blocks had changed? If the shape of the blocks had not changed, then they ought to fit together. If the blocks would not fit, the proper conclusion to draw would be that the gravitational field had distorted them. The experiment could not give any information about the geometry of the space itself. Empty space cannot be measured; we can only measure bodies and their distances apart. In empty space there is nothing to fix the shape or position of geometrical figures. They can be constructed in any way desired, and measurements made of them relate to the shape of the figures and give no information about the space in which they exist.

The conception of a curvature or distortion of empty space is meaningless, since only material bodies can be measured. Of course it is possible to try to deduce the geometrical properties of space from observations on material bodies, but it seems doubtful whether the results of

such attempts are anything but the geometrical properties of the material bodies. Einstein writes:[1] "We entirely shun the vague word 'space,' of which we must honestly acknowledge we cannot form the slightest conception, and we replace it by 'motion relative to a practically rigid body of reference.' "

This difficulty may be removed by supposing that space is not empty, but filled with a medium, the ether, and we may suppose that the properties of the ether are such that particles (which may be merely modified ether or singularities in it) move in it along paths which are straight where the ether is uniform, and curved where the properties of the ether vary from point to point. Or we may simply say that the ether is space. Since it always fills up space, we have no experience of space except when full of ether, and so we may identify space with the ether. It seems to the writer better to retain the ether and not get rid of it by supposing that space as such can have geometrical or other properties.

A space in which a particle describes a curve must have some degree of substantiality, and it seems more in accordance with experience to say that it contains a medium having the necessary properties than to attribute these properties to a vacuum.

It is admitted that in different parts of space bodies move in different ways. It is clear, then, that different parts of space have different properties, and, therefore, physical properties are present in space. But the possession of physical properties is the only known attribute of matter, so that we are entitled, if we so choose, to regard space as filled with a medium. If we do not so choose, then we must regard space itself as having properties, and we ought to

[1] "The Theory of Relativity," authorized translation by R. W. Lawson, page 9.

conclude that matter does not differ essentially from space, for matter merely has physical properties. If so, then space is full of matter, so that we get back to the ether in this way also.

In recent years many physicists seem to have come to the conclusion that the ether does not exist, because it is found to be impossible to detect any kind of effect due to the motion of translation of the earth through it. The old arguments for the ether, however, are still valid. Gravitational and electromagnetic actions take place across a vacuum, and light travels through a vacuum and has all the properties to be expected of a wave motion through a medium. To give up the ether merely because it fails to manifest its presence in one particular way is absurd and never ought to have been suggested.

Einstein's principle of equivalence provides a good illustration of how something which is everywhere present may fail to produce any observable effect. On a material system falling freely in a uniform gravitational field there should be no observable effect due to the field, although the system may move in it with ever-increasing velocity. The acceleration is produced by the field, but is the same for all parts of the system and so does not change the relative motion of the parts of the system, and therefore produces no observable effects.

It seems that the general principle that any action on a material system which is equal on all parts of the system will produce no effect observable in the system, might be put forward as plausible. It is only differential effects, due to variations in the action over the volume of the system, which can be observed.

If a system is moving with uniform velocity through the ether which remains everywhere at rest, then if all parts of

the system are equally affected by the relative motion, it is not surprising that no effects observable in the system are produced. The effects produced exactly compensate the action, just as in the case of a uniform gravitational field. No observable effects should be expected and none are found. It is not a case of a surprising compensation of effects, but a perfectly natural state of things. Motion through the air or through water gives observable effects because the action is not uniformly distributed over all the parts of the system.

According to the principle of equivalence, the visible universe may, for all we know, be moving with an enormous acceleration due to a uniform gravitational field produced by enormous masses outside. Such a field can produce no observable effects, and so its presence or absence cannot be determined. We may say, if we like, that it does not exist for us.

In the case of the ether, no effect due to uniform motion through it can be detected, but its presence is manifested in other ways independent of uniform motion which have been mentioned. Its presence and important functions need not be doubted. We have gravitational electric and magnetic fields which can exist together in so-called empty space, and it seems much more reasonable to say that such fields are modifications of a single medium, the ether, than to suppose that they exist in empty space or are due to modifications of empty space.

In what follows it will be supposed that there is a medium, the ether, filling up all space, and a theory of gravitation recently put forward by the writer[1] and based on this idea and on the electrical theory of matter will be described.

The atoms of matter, according to Rutherford, consist

[1] "Physical Review," January, 1921.

of a minute positively charged nucleus surrounded by a number of negative electrons, the positive charge on the nucleus being equal to the total negative charge on the electrons. This theory is confirmed in many ways and will be adopted here. An electrically neutral body is, therefore, a system of electrical charges, and its mass is probably wholly electromagnetic.

The facts which a theory of gravitation has to explain are three:

(1) Newton's law of gravitation.
(2) The deflection of light by the gravitational field of the sun.
(3) The change of frequency of the spectral lines emitted by atoms in the sun.

(1) Is not absolutely exact, according to Einstein.
(2) Was predicted by Einstein and verified by observations during the solar eclipse of 1919.
(3) Was predicted by Einstein and its existence is still in doubt, but the most recent results are favorable to it, so its truth will be assumed here.

If we assume matter to consist of electrical charges immersed in a medium, then, to explain Newton's law of gravitation, we must consider under what circumstances a neutral electrical system tends to move through a medium in which it is immersed.

It is well known that an electrostatic system immersed in a medium the specific inductive capacity (K) of which varies from point to point tends to move in the direction in which K increases most rapidly. It is natural, therefore, to test the suggestion that gravitation may be due to variations of the specific inductive capacity of the ether. If we find that the three facts given above can all be explained on

such a theory, we shall have good reason to consider the theory to be a possible one.

The observed deflection of light passing by the sun requires that the refractive index (n) of the ether be equal to $1 + 2\,m/r$, where m denotes the sun's mass and r the distance from his center. The distance r is expressed in units of length equal to the distance traveled by light in unit time, and the mass m is expressed in units of mass which produce unit gravitational acceleration at unit distance. If the unit of time is one second, then the unit of length is 3×10^{10} cms. and the unit of mass is about 4×10^{38} grams. The refractive index of the ether is equal to $\sqrt{\mu K}$, where μ is its magnetic permeability and K its specific inductive capacity. Hence, if we assume $n = 1 + 2\,m/r$, we have $(\mu K)^{\frac{1}{2}} = 1 + 2\,m/r$. The ratio m/r is always very small. Its greatest value in the solar system is about 2×10^{-6} at the surface of the sun. The equation $(\mu K)^{\frac{1}{2}} = 1 + 2\,m/r$ can be satisfied by taking $\mu = 1 + 2\,m/r$ and $K = 1 + 2\,m/r$.

The force on a neutral electromagnetic system in a medium in which it can move freely due to variations in K and μ is equal to $\frac{1}{8\pi}\left\{E^2\nabla K + H^2\nabla\mu\right\}$ per unit volume. Here E is the electric intensity and H the magnetic intensity. The electrical energy per unit volume is $KE^2/8\pi$, and the magnetic energy is $\mu H^2/8\pi$. Let W denote the electrical energy and W^1 the magnetic energy per unit volume. Then the expression for the force becomes

$$F = W\nabla K/K + W^1\nabla\mu/\mu$$
$$= W\nabla \log K + W^1\nabla \log \mu.$$

If $\mu = 1 + 2\,m/r$ and $K = 1 + 2\,m/r$, then neglecting squares of m/r $\log K = 2\,m/r$ and

$$F = -2\,m\,(W + W^1)/r^2.$$

The electromagnetic mass m^1 of the system in the units we

are using is equal to $\int (W + W^1) d\sigma$, where $d\sigma$ denotes an element of volume, so that for a small system, in which r can be regarded as constant,

$$F_r = -2\, mm^1/r^2$$

where F_r is now the total force on the system. This is just twice the Newtonian attraction $-mm^1/r^2$.

It appears that if we assume for μ and K values which give the observed deflection of light, then we get the gravitational attraction twice too big.

So far we have supposed that the electromagnetic system remains unchanged in size when μ and K vary, but since the forces between the parts of the system depend on μ and K, we should expect the size to vary.

If l is a quantity proportional to the linear dimensions of the system and equal to unity when $K = \mu = 1$, then the electrical energy of the system can easily be shown to be inversely as Kl and the magnetic energy inversely as μl, so that the force on the system, per unit volume, is given by

$$F = W \nabla \log Kl + W^1 \nabla \log \mu l.$$

Hence to get the observed gravitational attraction, we must have

$$\mu l = Kl = 1 + m/r \text{ or } l = 1 - m/r.$$

We should expect the dimensions of such a system to be determined by the dimensions of the electrons it contains. The size of an electron may be regarded as determined by an equilibrium between the tension inside it and the stress in the electric field outside it. The internal tension may be regarded as a sort of elastic reaction against the electric displacement at its surface. We should therefore expect the tension to be proportional to the displacement. This gives

$$\frac{1}{Ka^4} \propto \frac{1}{a^2}$$

where a is the radius of the electron, so that $aK^{\frac{1}{3}}$ is constant. If $K = 1 + 2\,m/r$, this gives $a \propto 1 - m/r$.

If the linear dimensions of the system are proportional to a, we get, therefore, $l = 1 - m/r$, which, as we have just seen, is the value required to give the observed gravitational attraction.

Thus it appears that if we assume $\mu = 1 + 2\,m/r$, and $K = 1 + 2\,m/r$, then we can explain the deflection of light by the sun and the gravitational attraction. The contraction given by $l = 1 - m/r$ is in agreement with Einstein's theory.

The third fact, that the frequency of the light emitted by an atom on the sun should be diminished by the gravitational field, can also be easily deduced from the present theory.

According to Bohr's theory of spectral lines, the frequency is proportional to the energy emitted when the atom passes from one stationary state to another. If the atom is in ether for which $K = 1 + 2\,m/r$ instead of $K = 1$, its energy in each of its stationary states will be diminished in the ratio $1 - m/r$, and so the frequency of any line which it emits will be diminished in the same ratio exactly as predicted by Einstein.

Thus it appears that the theory proposed leads to a simple explanation of all the three facts of gravitation, and so must be regarded as a possible theory.

The squares and higher powers of the ratio m/r have been neglected, so that the results obtained are only accurate to the first order of m/r. To this order the results obtained agree with Einstein's theory, so that the theory here proposed may be regarded as a physical interpretation of his theory.

The only sort of physical theory which seems possible is one depending on the physical properties of the ether, and

μ and K are the only properties of the ether with which we are acquainted. The nature of these properties is unknown, so that we have only explained the facts of gravitation in terms of unknown quantities, and we do not know why μ and K should be changed near matter in the way assumed.

Einstein's theory has the very great advantage that it is based on general principles derived from experience, whereas the theory just described depends on the assumption of a variation of K and μ, chosen so as to give results in agreement with the facts. The present theory has the advantage that it is very simple and requires no elaborate analysis; it may prove useful by enabling new phenomena to be predicted.

H. A. Wilson.

IV

THE INTEGRAL AND ITS GENERALIZATIONS

THE definite integral can be considered from several points of view. In college text-books it is defined for functions of simple type (continuous and with a finite number of maxima and minima) as the limit of a certain sum; but in its applications to mechanics it is expressed frequently in a form which does not correspond rigorously to the definition. A moment of inertia, for example, may be defined in the form $\int r^2 dM$, where M is not equal to r, nor is r a function of M. The idea of integral is broader than the elementary type.

It is of some interest to consider the broader notion, to analyze it and reduce it as far as possible to a logical form amenable to the methods of mathematics. For this purpose it is helpful to consider some of the less mathematical and less logical concepts which are felt to be akin to integrals. The first to occur to an applied mathematician is that of a moment. We have a fairly clear "a priori" idea of what is meant by the moment of a mechanical structure about some axis. It bears the same relation to rotation as does a force to parallel motion. It can easily be measured experimentally. The same is true of moment of inertia or rotational inertia. A similar idea, though not so simple, is that of the total potential due to an electrostatic or gravitational system.

In all these cases we have the familiar space and time as a background. In a more general and vaguer field of thought there is the average. Statistical averages are obtained for many kinds of entities, numerically measurable

by necessity, but entities dependent on objects or qualities which are not always numerical in their essence. These averages are, in practice, merely ratios of sums of a finite number of terms, but there is a feeling in many problems that there is a ghostly "correct" average to which we are but approximating by our rough methods. It is true that in some cases this ghost is misleading and illogical, as in the case of mortality statistics (unless we are considering the biological problem of the average mortality of all living tissue). Nevertheless one feels frequently that if it could only be grasped there is a limiting true average even in problems and sciences in which space and time or even pure number enter in a secondary manner, in which these appear as conditions rather than as fundamental characteristics. In mathematics itself the modern tendency is to consider not only collections of numbers and relations between numbers, but collections of functions, of curves, of logical classes, and so on. And there appears no absolute objection to such ideas as that of the average maximum of a collection of functions or the average area of a collection of closed curves, except that these concepts are vague and had not until recently been defined in a satisfying form. We shall see that at the present time this objection has fallen to the ground.

How can we characterize this broader concept of integral? From the examples given and others which the reader may invent there emerge several characters which are universally found in any concepts which are akin to integrals or moments or averages.

There must be a background of entities which may be denoted by certain marks. In a simple average or in the moment of a finite set of particles these marks may be suffixes 1, 2, 3, .. which distinguish one individual from another. In the moment of a continuous structure the posi-

tive integers may not be sufficient. The number of "individuals" may be too great. We therefore denote the general mark by a noncommittal symbol "p," which, for imaginative purposes, can be thought of as a "point" in some general geometry. A number or a point in space, a function or a curve, a color or a quality, may be such a mark.

The collection of these marks considered in the particular case is called P_0. There is again a numerical property of these marks, determinate when a particular mark p is chosen, that is to say, a function of p denoted by f(p). For a moment of inertia p is a particle or material point, f(p) is the square of the distance of that point from the axis. For an average error, p stands for a particular case considered, f(p) for the error involved in that case. For an ordinary integral the mark p is a number x lying in an interval a to b and f(p) = f(x) is some function of x. For a functional space p is a function belonging to a class P_0, f(p) is a functional of the function p.

In the case of closed curves p may be a closed curve of some simple type, f(p) the Green's function for that curve and two fixed points A, B. [A Green's function is usually regarded as a function of one variable point for a fixed curve and fixed point in the enclosed area, but it is convenient at times to hold the two points fixed and vary the curve.[1]]

The next step is, somehow, to sum f(p) over all marks p belonging to P_0. If the number of marks is limited we have an ordinary algebraic sum; if the collection P_0 has the power of the denumerable the sum is an infinite series; if P_0 is an interval in a number-space of a finite number of dimensions the sum is an integral. If P_0 is a continuous material body the sum may be a total moment or total potential at a fixed point.

[1] G. C. Evans, "Cambridge Colloquium," New York, 1918, p. 15.

But the summation cannot take place as a simple sum of the values of $f(p)$ for all p belonging to P_0 if P_0 has a power greater than the denumerable, because that would, in general, lead to an infinite result. It is necessary to proceed by a process of approximation. P_0 is subdivided into a finite number of subgroups $P_1, P_2, \ldots P_n$, to each of which some weight is assigned. In each subgroup P_r, a typical mark p_r is chosen and the value $f(p_r)$ of f at p_r is determined and multiplied by the weight m_r of the group. These products are added in the form

$$S_1 = m_1 f(p_1) + \ldots + m_n f(p_n).$$

A further subdivision is made with the result S_2 and so on. The sequence $S_1, S_2, \ldots$ may approach a limit which may be the same under many different processes of subdivision of P_0 and many different methods of choice of the representative of each group. The limiting sum is then called the integral of $f(p)$ over P_0 with the weighting system $m(P)$, P being a variable subgroup of P_0.

The integral depends, we see, not only on the existence of a function $f(p)$, but also on a method of weighting $m(P)$ a function of collections P of the marks p, picked out from P_0.

We now introduce a notation for our integral in order to be able to speak of it. Since it is a number which is determinate when the function $f(p)$ (p = element of P_0) and the weighting $m(P)$ (P = subgroup of P_0) are given, we call it

$$\int f(p)dm(P),$$

or $\int f dm$, or, when we are not so interested in the weighting, $S(f)$.

If there is to be any analogy with mechanics or statistics, the weighting $m(P)$ must be such that the sum of the weights of any number of distinct collections $P_1, P_2, \ldots P_n$ is equal

to the weight of the group consisting of all elements belonging to any one of them, that is, if $P = P_1 + P_2 + \ldots + P_n$,

$$m(P) = m(P_1) + m(P_2) + \ldots + m(P_n).$$

Now in such a subject as this, limiting processes and infinite series will naturally occur and difficulties will constantly appear unless we allow the weighting to be additive even for an infinite set of subgroups. That is to say, when

$$P = P_1 + P_2 + \ldots,$$
$$m(P) = m(P_1) + m(P_2) + \ldots,$$

provided there is no p common to two groups P_n.

Integrals which do not satisfy this relation are useful in some fields, but each requires its special treatment. Here we confine our attention to cases in which the additive property holds for an infinite as well as for a finite sum.

Now it happens that the sum of a set of classes or collections is independent of the order of the summation; for, by its definition, the sum of a set of classes is the class of elements belonging to some one class of the set. It follows that the sum $m(P_1) + m(P_2) + \ldots$, which is a numerical series, must be independent of the order of its terms. In the usual case the weights m(P) are essentially positive (or zero), and then the above series has the same sum in any order if it is convergent at all. But in electrostatics the charge in a volume can be regarded as a weight of the collection of points contained in the volume, and it will satisfy all the usual requirements for a weight except that of constant sign. To be as general as possible we should allow the possibility that m(P) is sometimes negative. Then the series $m(P_1) + m(P_2) + \ldots$ must be absolutely convergent.

It is of considerable value to distinguish in some way integrals which are based on essentially positive weight from the more general type, in the first place because the former are more common in practice and satisfy more completely

the properties of ordinary integrals; secondly, because, as we shall show, the other apparently more general sum can be regarded as the difference of two integrals with positive weighting. For an integral with positive weighting we use the symbol I(f) instead of S(f).

Returning to our attempted definition of the integral in terms of the weights, we have

$$S(f) = \lim S_n,$$

$$S_n = \sum_{i=1}^{i_n} f(p_{i,n})m(P_{i,n}).$$

It follows that if, for every p in P_o,

$$f(p) = f_1(p) + f_2(p) + \ldots + f_n(p),$$
$$S(f) = S(f_1) + S(f_2) + \ldots + S(f_n).$$

Again as with the weight m(P) so here with the function f, our integral will be more amenable to analysis if it satisfies this additive law not only for a finite but also for a denumerably infinite set of functions; that is, if when

$$f(p) = f_1(p) + f_2(p) + \ldots,$$
$$S(f) = S(f_1) + S(f_2) + \ldots.$$

Now, since S(f) is defined by means of a limit, this statement involves the validity of a change in order of a double limit. Again this double limit can be expressed as a double series in the following manner:

In the process by which S_n is obtained we subdivide P_o successively into parts so that $P_{i,n}$ is further subdivided into a set of distinct collections $P_{j,n+1}$ where j takes on a set of values depending on i. Or, again, $P_{i,n}$ itself is a part of one of the collections $P_{j,n-1}$ of the previous stage. Let

$$T_o(f) = f(p_{1,o})m(P_o),$$

$$T_n(f) = \sum_{i=1}^{i_n} [f(p_{i,n}) - f(p_{j,n-1})]m(P_{i,n}), \quad n = 1, 2, \ldots.$$

Then

$$S(f_r) = \sum_{n=o}^{\infty} T_n(f_r),$$

$$S(f_1) + S(f_2) + \ldots = \Sigma_r \Sigma_n T_n(f_r),$$
$$S(f) = \Sigma_n T_n(f) = \Sigma_n \Sigma_r T_n(f_r).$$

Such a change of order of summation in a double series we know to be legitimate if the double series is absolutely convergent. This suggests the restriction of the series $f_1 + f_2 + \ldots$ to absolutely convergent series.

We are now in a position to state the following general property of our integral:

If $f_1(p) + f_2(p) + \ldots$ is absolutely convergent and equal to $f(p)$ for all marks p in P_o, then

$$S(f) = S(f_1) + S(f_2) + \ldots .$$

Since the sum $f_1 + f_2 + \ldots$ is the same under any derangement of the terms, the sum $S(f_1) + S(f_2) + \ldots$ will be equal to $S(f)$ for any derangement and the series is absolutely convergent.

To distinguish this type of general integral from other types, for example that of E. H. Moore,[1] we call it an absolute integral.

Our problem is now explained in general terms. It remains to give it a more detailed and more logical examination.

Given a body of definitions it would be possible to examine the whole as a logical system for the consistency and independence of its postulates; but before this can be done it is necessary to determine some process of definition.

At first there may only be a small class of functions for which the initial defining processes, such as the method of subgroups and their representatives outlined above, give a single result. The analysis must be developed by stages proceeding from a smaller to ever wider classes of functions for which the integral is defined. And this extension will be

[1] E. H. Moore, "Bulletin of the American Mathematical Society," Vol. XVII (1912), p. 334.

most naturally obtained through sums of absolutely convergent series. We shall also use the principle that when a function lies between two other functions whose integrals are defined and equal, then this common value is also the integral of the given function. This principle, however, can only be used in the case of the positive integral I(f).

Before we proceed any further we introduce two concepts which are extremely important in connection with integrals of absolute type.

To any collection P contained in P_9 there corresponds a function f(p) equal to 1 when p belongs to P, and to 0 when p does not belong to P. Given the collection P, f(p) is completely and uniquely determined. Vice versa if a function of this type (equal everywhere either to 1 or 0) is given, it determines the collection P of marks p for which it is equal to 1. For such a function the integral will equal the weight of the collection P. For example, in a statistical problem the weight of any subgroup is also the average of the numbers equal to 1 in the subgroup and to 0 for all other marks or subgroups. Hence to state the weighting of the integral is the same thing as to state the value of the integral for a certain simple class of functions.

To express the problem in this way, in terms of an initial class of functions, has several advantages. On the one hand we may prefer to use for the purposes of our exposition a different class of initial functions; for example, functions defined by certain series or polynomials; on the other hand the first step from these special functions to the next class may suggest the processes of extension to even wider classes.

The second concept is related to monotone sequences. If a series is absolutely convergent it is the difference of two convergent series of positive (strictly, non-negative) terms, in which the partial sums form monotone sequences. Mono-

tone sequences and absolute convergence are therefore intimately related and this implies that monotone sequences are essential items in the theory of integrals of absolute type. Now in order to obtain such sequences it is valuable, if not necessary, to consider the greater of two numbers.

Let A be the class of numbers not greater than the number a, B the class of numbers not greater than b, and so on. In logic there are two combinations of classes, the logical sum $A \vee B$ which consists of all numbers not greater than one of the two numbers, that is, not greater than the greater of a and b, and the logical product $A \wedge B$, which consists of all numbers not greater than either of the two numbers, that is, not greater than the lesser of a and b. This suggests the following symbols and the corresponding properties:

$$a\gamma b = \text{the greater of a and b},$$
$$= \tfrac{1}{2}(a+b) + \tfrac{1}{2}|a-b|.$$
$$a\lambda b = \text{the lesser of a and b},$$
$$= \tfrac{1}{2}(a+b) - \tfrac{1}{2}|a-b|.$$
$$-(a\lambda b) = (-a)\gamma(-b), \quad -(a\gamma b) = (-a)\lambda(-b),$$
$$(a\gamma b)\gamma c = a\gamma(b\gamma c), \quad (a\lambda b)\lambda c = a\lambda(b\lambda c),$$

if $a \leqq c$, $(a\gamma b)\lambda c = a\gamma(b\lambda c)$, and we have the right to use the notation $a\gamma b\gamma c$, $a\lambda b\lambda c$, and (when $a \leqq c$) $a\gamma b\lambda c$. Moreover

$$a\gamma b + a\lambda b = a + b,$$
$$a = a\lambda b + (a-b)\gamma 0.$$

Many other similar identities could be proved by inspection or by correlation with the logical classes A, B, ...

Again, if $a \leqq b$, $a\gamma c \leqq b\gamma c$, $a\lambda c \leqq b\lambda c$, and if $\lim a_n = a$, $\lim a_n\gamma c = a\gamma c$, $\lim a_n\lambda c = a\lambda c$.

We can extend the same notation to functions so that $f\gamma g$ is that function which for each p is equal to the greater of f(p) and g(p). The above equalities and inequalities will be immediately applicable to functions as well as real numbers. One point may be noticed, that if f_1 is the

function corresponding to the collection P_1 and f_2 to P_2, then $f_1 \gamma f_2$ corresponds to $P_1 + P_2$ and $f_1 \lambda f_2$ to $P_1 P_2$.

Now if an initial class of functions is given which possess integrals in such a way that the additive postulate is satisfied, then we can proceed to define the integral of any linear combination of these functions and this will yield consistent results. The chief difficulty consists in the next stage, which is not immediately obvious. One important requirement is that a distinctly dynamic and constructive process should be present in order that the integral may be defined for a far wider class of functions than the initial class. E. H. Moore has proceeded by considering limits of relatively uniform convergent sequences. This leads to an interesting development to which the reader is referred.[1] It is, however, a process distinctly different—as a generative process—from the processes invented by Borel, Lebesgue, and W. H. Young. We prefer to follow the latter development. In this there is maintained on the one hand a constant correlation with the weight of subgroups, and on the other hand with a type of absolute convergence. These two are related more closely than might at first be imagined. For just as a series Σu_n is absolutely convergent when and only when the series $\Sigma \mid u_n \mid$ is convergent, so in the theory of Lebesgue integrals, $f(x)$ is summable if, and only if, $\mid f(x) \mid$ is summable.

In the case where p is a real number between 0 and 1 let us assume, as an example to show the relations between the weights of intervals and absolute summability, that the initial class of functions is such that any linear combination of functions of the class and the modulus of any function of the class are members of the class. Suppose also that $f(x) = 1\ (0 \leqq x \leqq 1)$ and $f(x) = x\ (0 \leqq x \leqq 1)$ are among

[1] E. H. Moore, "Bulletin of the American Mathematical Society," Vol. XVII (1912), p. 334.

the members of the initial class. Then it will be possible to assign a weight to any interval. For the function

$$0 \gamma (mx - b) \lambda 1$$

is equal to 0 from $x = 0$ to $x = b/m$, thence linear to $x = (b + 1)/m$ where it is equal to 1 and remains equal to 1 up to $x = 1$. Since $f\gamma g$ is a linear combination of f, g, and $|f - g|$, $f\gamma g$ is a member of the initial class if f and g are members. Therefore the function above defined possesses an integral. Now make m and b increase indefinitely but in such a way that $(b + 1)/m$ remains equal to c $(0 \leqq c \leqq 1)$. The limiting function will be equal to 0 for $x < c$ and to 1 when $x \geqq c$. If the limiting process is valid the integral of this function can be regarded as the weight (measure) of the closed interval $c \leqq x \leqq 1$. Subtracting another interval $d < x \leqq 1$ (this time keeping b/m fixed and equal to d) we obtain the weight of any closed interval (c, d).

Reciprocally, if the original aspect is that of the weighting of subgroups, the set of marks p for which $|f(p)|$ has a given value is simply the sum of those for which $f(p)$ and for which $-f(p)$ have that value. Hence the determination of the integral of $|f|$ will be of the same type as for f and $-f$.

With this general somewhat diffuse and nonlogical analysis of the foundations performed we are in a position to make a careful attack of the problem.

We assume, first, that there are certain marks p belonging to a class P_o; secondly, that there is a class T_o of functions $f(p)$ of these marks defined for all p in P_o; thirdly, that this class is such that when f, f_1 belong to the class so also do $|f|$, cf (where c is any real constant), and $f + f_1$.

For each of these functions of the fundamental class T_o there is defined in some way an integral $S(f)$ which satisfies the following postulate:

(A) If $f_1(p)$, $f_2(p)$, ... form a finite or denumerable set of

functions belonging to T_o, if their sum is f(p) and the sum $|f_1| + |f_2| + \ldots$ is g(p) for every p in P_o, where f(p), g(p) are also members of T_o, then

$$S(f) = S(f_1) + S(f_2) + \ldots .$$

This postulate can be analyzed into at least three postulates.

(C) $S(cf) = cS(f)$, if c is a real constant and if f belongs to T_o.

(S) $S(f_1 + f_2) = S(f_1) + S(f_2)$, if f_1, f_2 belong to T_o.

(L) $\lim S(f_n) = 0$, if $f_1(p) \geqq f_2(p) \geqq \ldots \geqq 0 = \lim f_n(p)$ for every p in P_o where f_1, f_2, ... belong to T_o.

The postulate (A) implies these three together for (S) is a special case of (A), (L) is a special case if we substitute $f_n' = g - (|f_1| + |f_2| + \ldots + |f_n|)$ where

$$g = |f_1| + |f_2| + \ldots \text{ in (A).}$$

Finally (C) is a consequence of (S) when the constant is rational, and of this together with (A) when c is the sum of an absolutely convergent series of rational numbers, which is true for any real value.

But (A) implies more than is implied by (C S L) together. Let $f(p) \geqq 0$ for all p in P_o and consider some finite set of functions $\phi_1, \phi_2, \ldots \phi_n$ such that $\phi_i(p) \geqq 0$ for all p in P_o ($i = 1, 2, \ldots n$) and such that

$$\phi_1 + \phi_2 + \ldots + \phi_n = f.$$

Then, when the ϕ's are varied, postulate (A) implies that $|S(\phi_1)| + |S(\phi_2)| + \ldots + |S(\phi_n)| \leqq I(f)$ where I(f) depends only on f and not on the particular subdivision of f.

For, if not, given any integer s we could choose the ϕ's in such a way that

$$\phi_{1,s} + \ldots + \phi_{n_s,s} = f, \phi_{i,s} \geqq 0, i = 1, 2, \ldots n_s,$$

$$|S(\phi_{1,s})| + \ldots + |S(\phi_{n_s,s})| > 2^s,$$

or if $\psi_{i,s} = \phi_{i,s}/2^s$,

$$\psi_{1,s} + \ldots + \psi_{n_s,s} = f/2^s,$$

$$| S(\psi_{1,s}) | + \ldots + | S(\psi_{n_s,s}) | > 1.$$

Now if $f = \Sigma_s f/2^s$

$$= \Sigma_s \Sigma_{i=1}^{n_s} \psi_{i,s}$$

which can be expressed as a single series of positive terms which is absolutely convergent, then postulate (A) asserts that

$$\Sigma_s \Sigma_i | S(\psi_{i,s}) |$$

is convergent, in contradiction to our result that each partial sum of terms summed for a particular s is greater than 1. Hence there does exist such an upper bound I(f).

Let us express this inequality in a simpler form. If f is a member of T_0 (all our functions have so far been assumed to be of this class), $|f|$ is a member of T_0, and if $f_1 = f\gamma 0$, $f_2 = -f\lambda 0$,

$$f = f_1 - f_2, \quad |f| = f_1 + f_2.$$

Now f_1 and f_2 are non-negative and therefore by the analysis immediately preceding,

$$\begin{aligned} | S(f) | &= | S(f_1) - S(f_2) | \\ &\leqq | S(f_1) | + | S(f_2) | \\ &\leqq I(| f |). \end{aligned}$$

A further consequence of (A) is therefore

(M) $$| S(f) | \leqq I(| f |)$$

if f is a member of class T_0, where $I(\phi)$ is a non-negative number defined for all non-negative functions ϕ belonging to T_0 such that if $\phi \leqq \psi$, $I(\phi) \leqq I(\psi)$.

The latter part of this statement follows immediately from the definition of $I(\phi)$ as an upper bound (by upper bound we mean "least upper bound").

It will later be proved that the postulate system C S L M together imply (A) and therefore (A) and (C S L M) are logically equivalent.

Let us first consider positive integrals. A positive

integral is such that the weight of any subgroup P is non-negative. This implies the more general assertion that if $f(p) \geqq 0$ for all p in P_o, $I(f) \geqq 0$. This we can designate postulate P. Taken in conjunction with (S) or (A) it implies that if $f \leqq g$, $I(f) \leqq I(g)$; and this again that

$$| I(f) | \leqq I(| f |),$$

which is a special form of (M).

Then AP imply C S L P while C S L P imply C S L M. We can now prove that C S L imply A, so that the two latter systems are equivalent, and since (A) implies (M), (M) is not, after all, independent of C S L, but is really implied by them.

For let $f_1, f_2, \ldots$ be a denumerably infinite set of functions satisfying the conditions of postulate (A). Let

$h_n = g - (| f_1 | + | f_2 | + \ldots + | f_n |)$,

$k_n = (g - f) - [(| f_1 | - f_1) + (| f_2 | - f_2) + \ldots + (| f_n | - f_n)]$.

Then h_n, k_n satisfy the conditions of postulate (L) and if this is assumed, $\lim S(h_n) = 0$, $\lim S(k_n) = 0$. Combining these with C S we see that (A) is satisfied. Furthermore if c is a real constant, it is not difficult to prove that S L imply C. On the other hand S and L are independent, for if

$S(f) = \lim f(x)$ as x approaches 0 through positive values,

S is satisfied but not L in general; while if

$$S(f) = f^2(0),$$

L is satisfied but not S.

Again P is independent of L and S. Therefore we can write our postulates as an irreducible minimum in the form (A) for the general integral S(f), or in the equivalent form (S L), and then add for the positive integral I(f) the postulate (P) to either system. But it is helpful to remember that C, M are consequences of either system.

We shall show how the general S-integral can be expressed as the difference of two positive I-integrals, then consider

how the positive integral is extended and state a number of examples illustrating some applications of the general theory. After that we shall state some of the more important theorems which are needed in applications.

If $f(p) \geqq 0$, for all p in P_o and if f is of class T_o, define $I_1(f)$ as the upper bound of $S(\phi)$ for all functions ϕ of class T_o such that $0 \leqq \phi \leqq f$. This upper bound exists, since by M,

$$\begin{aligned} S(\phi) &\leqq I(|\phi|) = I(\phi) \\ &\leqq I(f). \end{aligned}$$

[At present there is no relation between I_1 and I, and neither has been proved to be a positive or I-integral.]

The function $\phi = 0$ is a member of T_o and its S-integral is 0, and therefore

$$I_1(f) \geqq 0,$$

so that I_1 satisfies postulate P.

Let f_1, f_2 be two non-negative functions of class T_o. If $0 \leqq \phi_1 \leqq f_1$, $0 \leqq \phi_2 \leqq f_2$, then $0 \leqq \phi_1 + \phi_2 \leqq f_1 + f_2$. However ϕ_1, ϕ_2 may be varied, and they can be varied independently,

$$\begin{aligned} S(\phi_1) + S(\phi_2) &= S(\phi_1 + \phi_2) \\ &\leqq I_1(f_1 + f_2), \\ I_1(f_1) + I_1(f_2) &\leqq I_1(f_1 + f_2). \end{aligned}$$

On the other hand, if $0 \leqq \phi \leqq f_1 + f_2$, $\phi - f_1 \leqq f_2$.

Now $\phi = \phi\lambda f_1 + (\phi - f_1)\gamma 0$

$= \phi_1 + \phi_2$,

where $0 \leqq \phi_1 \leqq f_1$, $0 \leqq \phi_2 \leqq f_2$. Hence

$$\begin{aligned} S(\phi) &= S(\phi_1) + S(\phi_2) \\ &\leqq I_1(f_1) + I_1(f_2). \end{aligned}$$

But this is true however we vary ϕ and therefore

$$I_1(f_1 + f_2) \leqq I_1(f_1) + I_1(f_2).$$

When these two inequalities are combined we see that $I_1(f)$ satisfies postulate S, at least if f_1, $f_2 \geqq 0$.

For any function of class T_o we define

$$I_1(f) = I_1(f\gamma 0) - I_1(-f\lambda 0).$$

If $f = g - h$ where g, h are non-negative, $g \geqq f\gamma 0$, $g = f\gamma 0 + k$, $h \geqq -f\lambda 0$, $h = k - f\lambda 0$, where $k \geqq 0$.
By the case already considered,

$$I_1(g) - I_1(h) = I_1(f\gamma 0) + I_1(k) - I_1(k) - I_1(-f\lambda 0)$$
$$= I_1(f).$$

By this means it is possible to show that postulate S is satisfied by all functions of class T_0.

Instead of proving L directly we shall prove A in the case where the functions summed are non-negative. Let

$$f_1 + f_2 + \ldots = f$$

where all these functions are of class T_0 and non-negative. Let ϕ be of class T_0 and such that $0 \leqq \phi \leqq f$. Take

$$\phi_1 = \phi\lambda f_1,\ \psi_1 = (\phi - f_1)\gamma 0,$$
$$\phi_2 = \psi_1\lambda f_2,\ \psi_2 = (\psi_1 - f_2)\gamma 0,$$

and so on.

Then $0 \leqq \phi_n \leqq f_n$, and $\phi = \phi_1 + \psi_1$,

$$= \phi_1 + \phi_2 + \psi_2,$$
$$= \ldots\ldots$$

Also $\psi_1 - f_2 \leqq \psi_1$, $0 \leqq \psi_1$, and therefore $\psi_2 \leqq \psi_1$. Similarly $\psi_3 \leqq \psi_2$, $\psi_4 \leqq \psi_3$, ...
Now $\phi - f_1 \leqq f - f_1$, so that $\psi_1 \leqq f - f_1$; $\psi_1 - f_2 \leqq f - f_1 - f_2$, so that $\psi_2 \leqq f - f_1 - f_2$, and so on. Then since $\psi_n \geqq 0$, $\lim \psi_n = 0$. Therefore

$$S(\phi) = S(\phi_1) + S(\phi_2) + \ldots + S(\phi_n) + S(\psi_n)$$
$$\leqq I_1(f_1) + I_1(f_2) + \ldots + I_1(f_n) + S(\psi_n).$$

But if we choose any functions $g_1, g_2, \ldots$ such that
$0 \leqq g_n \leqq f_n$, $g_1 + g_2 + \ldots + g_n \leqq f$, and

$$S(g_1) + S(g_2) + \ldots + S(g_n) \leqq I_1(f),$$
$$I_1(f_1) + I_1(f_2) + \ldots + I_1(f_n) \leqq I_1(f).$$

Therefore the series of non-negative terms

$I_1(f_1) + I_1(f_2) + \ldots$ is convergent and its sum is not greater than $I_1(f)$. To return, since $\lim S(\psi_n) = 0$,

$$S(\phi) \leqq I_1(f_1) + I_1(f_2) + \ldots,$$

and since ϕ can be varied at random this proves a second inequality which with the other proves that

$$(I_1(f) = I_1(f_1) + I_1(f_2) + \dots .$$

To prove L itself we substitute f_1 for f, f_2 for $f - f_1$, ... in the above. Then we have proved that $I_1(f)$ is an integral satisfying SL and it is also of positive type.

Now we define $I_2(f) = I_1(f) - S(f)$, and I_2 will also be an integral of positive type, for among the functions ϕ we can choose f itself in the definition of $I_1(f)$.

Then we also take $I(f) = I_1(f) + I_2(f)$ and this is a positive integral. I_1 is called the positive, I_2 the negative and I the modular integral associated with S(f). The relations between them are

$$S(f) = I_1(f) - I_2(f), I(f) = I_1(f) + I_2(f).$$

We have now proved that any S-integral is the difference of two positive integrals.

$$\begin{aligned} |S(f)| &= |S(f\gamma 0) - S(-f\lambda 0)| \\ &\leqq |S(f\gamma 0)| + |S(-f\lambda 0)| \\ &\leqq I_1(f\gamma 0) + I_2(f\gamma 0) + I_1(-f\lambda 0) + I_2(-f\lambda 0) \end{aligned}$$

since all these integrands are positive. But $I = I_1 + I_2$, $|f| = f\gamma 0 - f\lambda 0$ and

$$|S(f)| \leqq I(|f|).$$

If $f \geqq 0, \phi_1, \phi_2, \dots \phi_n \geqq 0$ and if

$$\begin{aligned} \phi_1 + \phi_2 + \dots + \phi_n &= f, \\ |S(\phi_1)| + |S(\phi_2)| + \dots + |S(\phi_n)| &\leqq I(\phi_1) + I(\phi_2) + \dots + I(\phi_n), \\ &= I(f). \end{aligned}$$

Again, given any positive e and given $f \geqq 0$, we can choose ϕ so that $0 \leqq \phi \leqq f$ and $S(\phi) > I_1(f) - e/2$.

If we choose $e < 2I_1(f)$, $S(\phi) > 0$ so that $|S(\phi)| = S(\phi)$.

$$\begin{aligned} |S(f - \phi)| &= |I_1(f) - S(\phi) - I_2(f)| \\ &\geqq I_2(f) - [I_1(f) - S(\phi)] \\ &> I_2(f) - e/2. \end{aligned}$$

Therefore

$$|S(\phi)| + |S(f-\phi)| > I_1(f) + I_2(f) - e$$
$$= I(f) - e.$$

From these two inequalities we draw the conclusion that this I(f) defined in terms of I_1 and I_2 is the same upper bound of $|S(\phi_1)| + |S(\phi_2)| + \ldots + |S(\phi_n)|$, where $\phi_1, \phi_2, \ldots$ are non-negative functions whose sum is f, as that upper bound which we called I(f) previously in our analysis of postulate (A).

In extending the definition of the integral to functions outside the initial class the natural step is to functions which are the sums of absolutely convergent series of functions of class T_0. Since such a series is the difference of two series of positive terms, and to the latter correspond increasing sequences it is simpler, as W. H. Young has shown, to consider the latter first.

If $f_1 \leqq f_2 \leqq \ldots$ is a non-decreasing sequence of functions of class T_0, the initial class, then $\lim f_n$ exists in any case if we allow $+\infty$ as a possible value, and we say that $\lim f_n = f$ is of class T_1. It follows that

$$I(f_1) \leqq I(f_2) \leqq \ldots,$$

if I is some positive integral, and $\lim I(f_n)$ exists if, again, we allow $+\infty$ as a possible value.

This limit has been shown by the author[1] to be independent of the actual sequence defining f and to depend only on the limit f itself. We have a right to call the limit $\lim I(f_n)$ the integral of f, provided it is finite. If this is the case we define $I(f) = \lim I(f_n)$ and say that f is a summable member of T_1. It can then be shown that this new integral also satisfies postulates SLP. If f is any function, we define the upper semi-integral of f, $\dot{I}(f)$ as the lower bound of I(g) for all functions g of class T_1 such that $g \geqq f$. In other

[1] P. J. Daniell, "Annals of Mathematics," Vol. XIX (1918), p. 279.

words, we strive to define the integral of f by means of numbers which it cannot exceed and call the least possible of such upper boundaries for its value a semi-integral until we can ascertain that the integral itself is definite and unique. At the same time we define the lower semi-integral of f, $\underline{I}(f)$ as the negative of the upper semi-integral of $-f$. If it happens that the two semi-integrals are equal then f is said to be summable and its integral is defined to be the common value of the two semi-integrals.

The class of summable functions has the same properties as the initial class T_0. Indeed if f, f_1 are summable so are $|f|$, cf where c is a real constant, and $f + f_1$. Also the postulates SLP and with them A and C are satisfied where in A it is necessary, in general, to retain the condition that $|f_1| + |f_2| + \ldots$ is not greater than some summable function, such as a summable member of T_1, though any summable function serves the same purpose and gives no greater generality.

The fundamental theorem of this class of integrals is that if $f_1, f_2, \ldots$ is a sequence of summable functions converging to a limit f, and if a summable function g exists such that $|f_n| \leqq g$ for all values of n, then f is summable and $\lim I(f_n) = I(f)$.

Another interesting property which suggests another method of development is that the necessary and sufficient condition that f be summable is that, given any positive e, it is possible to find a function f_e of the initial class T_0 such that

$$\bar{I}(|f - f_e|) < e.$$

In other words, a summable function must be within any "distance" however short of some member of T_0, measuring "distance" in the function space by means of the upper semi-integral of the modular difference of the functions. This

geometrical analogy is very fruitful but cannot here be entered into.

For the general definition of S(f) we take

$$S(f) = I_1(f) - I_2(f),$$

provided f is summable with respect to both I_1 and I_2, which will happen whenever f is summable with respect to I the modular integral associated with S.

Sometimes an operation J(f) is given in some way for an already wide class of functions. It is an important problem to decide whether it coincides with an integral I(f) as defined above by extension. Evidently, in the first place, the two must be identical for members of the class T_0. If J(f) satisfies also, for all functions f for which it happens to be defined, the inequality

$$J(f) \leqq J(g), \text{ when } f \leqq g,$$

then it is sufficient to prove the identity of J(f) and the extension of I(f) for all summable members of class T_1. Or, what is the same thing, that when f is the limit of a non-decreasing sequence of functions f_n of class T_0,

$$J(f) = \lim J(f_n).$$

Again, since, of necessity, $\lim J(f_n) \leqq J(f)$, f_n being not greater than f, it is sufficient to show that $J(f) \leqq \lim J(f_n)$.

Illustrative examples.

(1) Absolutely convergent series. Let P_0 be the class of positive integers 1, 2, 3, ... with weights $w_1, w_2, \ldots$ assigned of such a character that $|w_1| + |w_2| + \ldots$ is convergent. Let f_n be a function of the integral variable n. Then in some cases, for example when f_n is uniformly limited, the series

$$S(f) = f_1w_1 + f_2w_2 + \ldots$$

will be absolutely convergent.

By what is known of absolutely convergent double series postulate A is satisfied. In this case we can define S(f) without the necessity of proceeding by successive stages.

However we could let T_o be the class of functions f_n equal to a number other than 0 for a finite set of values of n and 0 for all the others. A function f_n which is non-negative for all sufficiently large values of n would then be of class T_1.

$I_1(f)$ will be the sum of $f_n w_n$ for all non-negative w_n and $I(f)$ the sum $\Sigma_n f_n \mid w_n \mid$.

(2) Let P_o be the class of real numbers, x, where $0 \leqq x \leqq 1$. Let weighting be assigned to each interval equal to its length.

For T_o we choose the class of step-functions f(x) of the type

$$f(x) = c_i, \; x_{i-1} \leqq x \leqq x_i,$$

where (0, 1) has been divided into some finite set of intervals

$$0 = x_o < x_1 < \ldots < x_n = 1.$$

If we wish it is sufficient, and leads finally to no loss in generality, to confine T_o to be the class of functions of this type restricting the division points $x_1, x_2, \ldots x_n$ to be those obtained by dividing the interval (0, 1) into $n = 2^r$ equal parts, r being some integer. We then define

$$I(f) = \int f(x)dx = \Sigma_i c_i(x_i - x_{i-1}).$$

This integral is of positive type, for when $f(x) \geqq 0$, every $c_i \geqq 0$ and $I(f) \geqq 0$. This is the usual integral when the integrand f is sufficiently restricted. The extension given in this paper leads to the Lebesgue integral, as W. H. Young has shown. In place of the step-functions the class T_o could be taken to be the class of continuous functions, but then the proof of the existence of the integral is not immediate as it is for step-functions.

(3) To discuss the generalization of the ordinary integral, let P_o and T_o be as before and consider a definition of the general integral in the form

$$S(f) = \Sigma_i c_i[\alpha(x_i) - \alpha(x_{i-1})]$$

where α (x) is some function of x, $0 \leqq x \leqq 1$.

Postulate S is satisfied but not L necessarily. For example let

$$f_n(x) = 1, c_n \leqq x < d,$$
$$= 0 \text{ otherwise}$$

where c_n is made to approach d from below as n increases indefinitely.

The conditions $f_1(x) \geqq f_2(x) \geqq \ldots \geqq f_n(x) \geqq \ldots \geqq 0 = \lim f_n(x)$ for all x, are satisfied. But

$$S(f_n) = \alpha(d) - \alpha(c_n),$$
$$\lim S(f_n) = \alpha(d) - \alpha(d - 0).$$

Unless $\alpha(x)$ is continuous this difference may not be 0 as the postulate L demands. Either we must restrict $\alpha(x)$ to be continuous or reconstruct our definition. Let us study the problem rather from the point of view of weight. If we allow weight (measure) to be discontinuous it becomes necessary to distinguish between closed and open intervals, it is necessary to consider the weight of even a single point. As a matter of fact the problem can be solved satisfactorily if we let T_0 be the class of functions which are constant ($= c_i$) over each of a finite set of intervals Δ_i into which (0, 1) can be divided, if we define

$$S(f) = \Sigma_i c_i m(\Delta_i),$$

where $m(\Delta)$ is the measure or weight of the interval Δ. For this we define

$$m(\Delta) = \alpha(d+0) - \alpha(c-0), \Delta \text{ is } (c \leqq x \leqq d), \text{ closed;}$$
$$= \alpha(d+0) - \alpha(c+0), \Delta \text{ is } (c < x \leqq d)$$
$$= \alpha(d-0) - \alpha(c+0), \Delta \text{ is } (c < x < d), \text{ open;}$$
$$= \alpha(d-0) - \alpha(c-0), \Delta \text{ is } (c \leqq x < d).$$

By $\alpha(c+0)$ we mean the limit (assumed existent) of $\alpha(c+\epsilon)$ as ϵ approaches 0 through positive values. Similarly for $\alpha(c-0)$.

If in particular $\alpha(x)$ is a non-decreasing function of x, denoted by $\beta(x)$, then the above limits exist and all the postulates can be proved. In this case $M(\Delta) \geqq 0$, and the integral is an I-integral

$$I(f) = \int f(x)dm(\Delta)$$
$$= \int f(x)d\,\beta(x).$$

Again if $\alpha(x)$ is the difference of two non-decreasing functions $\beta_1(x)$, $\beta_2(x)$ the integral using α can be defined as the difference of the integrals using β_1, β_2,

$$\int f(x)d\,\alpha(x) = \int f(x)d\,\beta_1(x) - \int f(x)d\,\beta_2(x).$$

It is known that if $\alpha(x)$ is of limited variation, that is, such that $\Sigma_i \mid \alpha(x_i) - \alpha(x_{i-1}) \mid$ is limited by a number K independently of the method of subdivision, then $\alpha(x)$ can be expressed as the difference of two non-decreasing functions. The converse is easily proved since if $\alpha(x) = \beta_1(x) - \beta_2(x)$,

$\mid \alpha(x_i) - \alpha(x_{i-1}) \mid \leqq \beta_1(x_i) - \beta_1(x_{i-1}) + \beta_2(x_i) - \beta_2(x_{i-1})$ and the number K may be chosen to be $\beta_1(1) - \beta_1(0) + \beta_2(1) - \beta_2(0)$. The least function $\beta_1(x) + \beta_2(x) - \beta_1(0) - \beta_2(0)$ is called the total variation function $\omega(x)$ corresponding to $\alpha(x)$. This case leads to the Stieltjes integral with respect to a function of limited variation

$$\int f(x)d\,\alpha(x).$$

The first extension of the definition leads to integrals of continuous functions f(x), which are the Stieltjes integrals proper. Further extensions of the class of integrands on the lines of the Lebesgue integral lead to the general Radon-Young integral.[1] A frequently more useful notation is

$$\int f(x)dm(e)$$

where m(e) is an additive (for a denumerable infinity as well as for a finite number) function of sets e of values of the mark $p = x$. And then the corresponding modular integral can be expressed, after Radon, as

$$\int f(x) \mid dm(e) \mid.$$

1 J. Radon, "Sitzungsberichte der Akademie der Wissenschaften, Wien" (1913), p. 1295.

W. H. Young, "Proceedings of the London Mathematical Society," Vol. XIII (1914), p. 109.

(4) Let P_o be the set of all real numbers, T_o the class of functions f(x) equal to a constant other than 0 only over a finite number of bounded (compact) intervals, and 0 elsewhere. For the sake of simplicity choose $\beta(x)$ as a function which is non-decreasing from $-\infty$ to $+\infty$ and which may or may not approach a finite limit as x approaches $+\infty$ or $-\infty$. As in case (3) we may define the weight of a bounded interval Δ as

$$m(\Delta) = \beta(d+0) - \beta(c-0)$$

for a closed interval (c, d) and in a similar manner for other types of interval. Then for a member of T_o define

$$I(f) = \Sigma_i c_i m(\Delta_i)$$

where c_i is the constant value of f over the subinterval Δ_i. If it happens that $\beta(-\infty)$, $\beta(+\infty)$ both exist as finite numbers, the remaining analysis differs in no essential from that of case (3). If, however, $\beta(-\infty) = -\infty, \beta(+\infty) = +\infty$, for example, it appears that only functions f(x) which approach 0 with more than a certain rapidity of convergence at $\pm\infty$ will be summable. It still holds that f is summable when and only when | f | is summable. Cases in which

$$\int_{-\infty}^{\infty} f(x)\,d\beta(x)$$

is conditionally convergent will not appear as direct cases of our general integral, but will require separate handling as limiting cases beyond our immediate aim. For example, it will be possible to consider

$$\int_{-\infty}^{\infty} f(x)\,dx$$

as a case of an absolute integral if the functions satisfy a relation of type

$\lim | f(x) / x^{\lambda} | = 0$, as | x | increases indefinitely for some $\lambda > 1$.

But an integral like

$$\int_{-\infty}^{\infty} \sin x \, dx / x$$

will need separate treatment.

(5) Let P_0 be the class of real numbers, x, such that $0 \leqq x \leqq 1$, let T_0 be the class of step-functions constant over each of a finite set of subintervals, but restricted to be 0 in a neighborhood of $x = 0$ and in a neighborhood of $x = 1$. Let $\beta(x)$ be a non-decreasing function of x which may be $-\infty$ at $x = 0$ and $+\infty$ at $x = 1$. For any interval Δ not containing nor abutting on 0 or 1, define

$$m(\Delta) = \beta(d+0) - \beta(c-0)$$

for a closed interval $\Delta = (c, d)$ and similarly for the other types of interval. Define

$$I(f) = \Sigma_i c_i m(\Delta_i),$$

where $f(x) = c_i$ on the interval Δ_i. In particular, let $\beta(x) = -\operatorname{ctn} \pi x$. Then, provided f(x) is measurable in the sense of Borel and approaches 0 with sufficient rapidity at $x = 0, x = 1$, it will be possible to define

$$I(f) = \int_0^1 f(x) d\,\beta(x).$$

In the particular case given it is sufficient if

$$| f(x) / \sin^2 x |$$

is uniformly limited in the interval $(0 < x < 1)$ and if $f(x) = 0$ at $x = 0, x = 1$.

(6) Let T_0 be the class of real numbers $-1 \leqq x \leqq 1$, let T_0 be the class of functions constant over each of a finite set of subintervals but restricted by the condition that no endpoints of intervals are at $x = 0$. In other words, $x = 0$ is contained strictly within one of the intervals. For a particular example let $\alpha(-1) = \alpha(1) = 0$, and let α increase from either point towards $x = 0$ in such a way that $\lim [\alpha(\epsilon) - \alpha(-\epsilon)]$ exists as ϵ approaches 0.

If f(x) is continuous and if as x approaches 0, f(x) approaches f(0) with sufficient rapidity, f will be summable. For example, take

$$\alpha(x) = (1 - x^2)\sqrt{1 + 1/x + 1/x^2}.$$

Near $x = 0$, $\alpha(x) = (1/|x|)(1 - x^2)(1 + \frac{1}{2}x + \dots)$

$\alpha(\epsilon) - \alpha(-\epsilon)$ tends to the limit 1.

By considering $d\alpha/dx$ it is possible to prove that α increases from $x = \pm 1$ to $x = 0$.

In this particular example it is sufficient that f(x) be continuous at $x = 0$ to an order greater than 1. That is, that $|f(x) - f(0)| < K|x|^{\lambda}$, $\lambda > 1$, K independent of x.

For $\phi(x) = f(0)$ for all x is summable from -1 to 1 or in any subinterval enclosing the point 0 strictly, while $\psi(x) = f(x) - f(0)$ is summable throughout any subinterval whatever. (The term summable means here summable with respect to the particular integral defined, not ordinary Lebesgue summability.)

(7) In three dimensions an interval Δ will be the collection of points whose coördinates satisfy inequalities of the type

$$a_1 \leqq x \leqq a_2,\ b_1 \leqq y \leqq b_2,\ c_1 \leqq z \leqq c_2.$$

Some or all of these signs of inequality may be replaced by $<$. Then given a function $\alpha(x, y, z)$ we may define

$$\begin{aligned} m(\Delta) = {} & \alpha(a_2 + 0, b_2 + 0, c_2 + 0) - \alpha(a_2 + 0, b_2 + 0, c_1 \\ & - 0) - \alpha(a_2 + 0, b_1 - 0, c_2 + \\ & - \dots + \alpha(a_2 + 0, b_1 - 0, c_1 - 0) + \dots - \alpha(a_1 - \\ & 0, \beta_1 - 0, c_1 - 0). \end{aligned}$$

If $\alpha(x, y, z)$ is such that $m(\Delta) \geqq 0$ for all intervals Δ, the integral is of positive type. If α is the difference $\beta_1 - \beta_2$ of two functions of positive type we can define

$$S(f) = \int f dm_1 - \int f dm_2$$

and obtain a three-dimensional Stieltjes integral. Evidently more general cases can be considered by combining and

transforming the cases already given. In particular, we can extend (7) to any finite number of dimensions. It is only necessary to have some foundation in an additive function of intervals Δ. Again examples (1) to (6) could be generalized to several dimensions.

(8) This example due to G. C. Evans[1] suggests a different form of application of the general theory of integrals and of additive functions of sets. Let f(e) be an additive function of plane sets, that is, additive for an infinite sum as well as for a finite sum. Then a function of curves F(s) can be defined so that if s_1 is a particular curve,

$$F(s_1) = -\int_{\Sigma} \psi(P)\, df(e)$$

where P is a point of the fundamental set Σ and where

$$\psi(P) = \int_{s_1} \frac{\cos nr}{r}\, ds_1,$$

nr being the angle between the inward drawn normal to s_1 and r, r being the vector P_1P drawn from a point P_1 on s_1. It is then shown that $F(s_1) = f(e)$ where e is the set of points within s_1, if $F(s_1)$ is a "continuous" function of curves. This means that in some cases a function of curves can be used as a weight of the sets of points within the curves.

(9) If $(x_1, x_2, \ldots x_n \ldots.)$ is a point in a space of a denumerable infinity of dimensions such that $0 \leqq x_i \leqq 1$ $(i = 1, 2, \ldots)$, then we can define the weight $m(\Delta)$ of an interval Δ such as

$$a_1 \leqq x_i \leqq 1 - b_1 \ (i = 1, 2, \ldots)$$

as equal to

$$m(\Delta) = \operatorname{Prod}_{i=1}^{\infty} (1 - b_1 - a_i),$$

an infinite product which may diverge to 0 or converge to a value not greater than 1.

[1] G. C. Evans, "Rendiconti della Reale Accademia dei Lincei," Vol. XXVIII (1919) pp. 262-5.

Then if f(p) is a continuous function of $p = (x_1, x_2, \dots)$ in the sense of Fréchet we can define

$$\int f(p)dm(e)$$

an integral of an infinite number of dimensions.[1] The author has also considered functions of limited variation in an infinite number of dimensions. This type of integral might possibly be useful in connection with probability of sets of functions defined by means of Fourier constants or by the coefficients of a series expression.

(10) Recently N. Wiener[2] has investigated the preliminary problem of weighting in general integrals and in his example (d) defines an integral in a space of continuous functions. Wiener proves that every bounded continuous functional is summable in accordance with his definition of an integral. Further papers on this subject are to be published soon.

This is but a beginning of a new field.

In a further paper[3] by the author it is proved that not only is the general integral S(f) expressible as a difference of two positive integrals, but that a function λ everywhere equal to 1 or -1 exists such that for all summable f

$$S(f) = I(\lambda f)$$

where I is the modular integral associated with S, provided that there exists at least one summable function $h > 0$ except at marks p for which every summable f vanishes. For the simple Stieltjes integral this means that we can find a function λ equal everywhere to ± 1 such that if f is summable with respect to the weighting m(e) then

$$\int f(x)dm(e) = \int f(x)\lambda(x) \mid dm(e) \mid.$$

[1] P. J. Daniell, "Annals of Mathematics," Vol. XX (1919), p. 281; Vol. XXI (1919), p. 30.

[2] N. Wiener, "Annals of Mathematics," Vol. XXI (1920), December.

[3] P. J. Daniell, "Annals of Mathematics," Vol. XXI (1920), p. 203.

The same theorem can be applied to any number of dimensions. The function λ corresponds to a derivative of the function of sets m(e) with respect to its modular function of sets

$$\int_e 1 \mid dm(e) \mid.$$

This suggests the further problem of generalized derivatives. If m(e) is an additive function of sets e and if M(e) is additive and positive, if also $m(e) = 0$ whenever $M(e)$ is 0, then we may expect to find a function D(p) summable with respect to M(e) such that

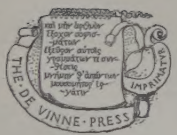
$$m(e) = \int_e D(p)dM(e).$$

At the same time it is to be expected that if f(p) is summable with respect to m(e) then

$$\int f(p)dm(e) = \int f(p)D(p)dM(e).$$

All this, however, is without rigorous justification, at present.

P. J. DANIELL.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